Subject :

LIMITS OF FUNCTION

Q: Describe the behaviour of the function

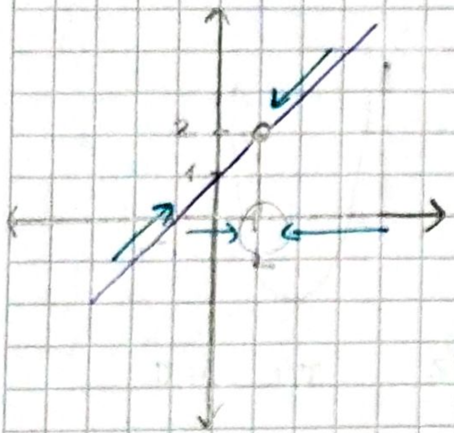
$f(x) = \dfrac{x^2-1}{x-1}$ near $x = 1$?

It is not defined at the point "+1"

$f(x) = \dfrac{x^2-1}{x-1}$

$\hookrightarrow x \neq 1$

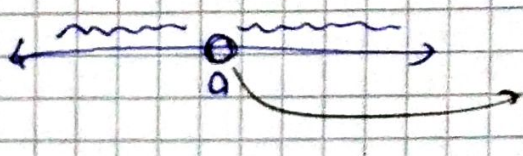
$f(x) = \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = (x+1)$



$\lim\limits_{x \to 1} f(x) = 2$ or $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = 2$

DEFINITION: If $f(x)$ is defined for all x near a, except possibly at a itself and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a but not equal to a, we say that the function f approaches the limit L as x approaches a and we write

$\lim\limits_{x \to a} f(x) = L$



not a, but close to a

Example: $\lim_{x \to 4} (x^2 - 4x + 1)$

Solution: $\lim_{x \to 4} (x^2 - 4x + 1) = 4^2 - 4.4 + 1$

$= \underline{\underline{1}}$

Bazen sadece $f(a)$ hesaplanarak da değerlendirilir. $f(x)$, $x = a$'yı kapsayan açık aralıkta tanımlanmışsa ve f (grafiği $(a, f(a))$ dan kesintisiz geçerse durum bu olacaktır.

Example:

a) $\lim_{x \to -2} \dfrac{x^2 + x - 2}{x^2 + 5x + 6} = ?$

$$\lim_{x \to -2} \frac{\cancel{(x+2)}(x-1)}{(x+3)\cancel{(x+2)}} = \lim_{x \to -2} \frac{x-1}{x+3}$$

$$= \frac{-3}{1} = -3 \,//$$

b) $\lim_{x \to a} \dfrac{\frac{1}{x} - \frac{1}{a}}{x - a} = ?$

$$\lim_{x \to a} \frac{\frac{a-x}{ax}}{\frac{x-a}{1}} = \frac{a-x}{ax} \cdot \frac{1}{x-a} = -\frac{1}{ax} = -\frac{1}{a^2} \,//$$

c) $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x^2 - 16} = ?$

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{(x-4)(x+4)} = \lim_{x \to 4} \frac{\cancel{\sqrt{x} - 2}}{\cancel{(\sqrt{x} - 2)}(\sqrt{x} + 2)(x+4)}$$

$$= \lim_{x \to 4} \frac{1}{(\sqrt{x}+2)(x+4)} = \frac{1}{4.8} = \frac{1}{32} \,//$$

Example:

a) $\lim_{x \to 3} \dfrac{x^2 - 6x + 9}{x^2 - 9}$

$$\lim_{x \to 3} \frac{\cancel{(x-3)}(x-3)}{\cancel{(x-3)}(x+3)} = \frac{3-3}{3+3} = \frac{0}{6} = 0$$

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b) $\lim_{x \to 2} \frac{x^4 - 16}{x^3 - 8} \Rightarrow \lim_{x \to 2} \frac{(x^2-4)(x^2+4)}{(x-2)(x^2+2x+4)}$

$\lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)(x^2+4)}{\cancel{(x-2)}(x^2+2x+4)}$

$= \frac{(2+2)(2^2+4)}{(2^2+2.2+4)} = \frac{4.8}{12} = \frac{8}{3}$

c) $\lim_{t \to 0} \frac{t^2+3t}{(t+2)^2-(t-2)^2} \Rightarrow \lim \frac{t(t+3)}{(t+2+t-2)(t+2-t+2)}$

$= \lim \frac{t(t+3)}{(2t)(4)} = \frac{t+3}{8} = \frac{0+3}{8} = \frac{3}{8}$

Example :

a) $\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = ? = \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)}$

$= \lim_{h \to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$

$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2} = \frac{1}{\sqrt{4+h}+2}$

$= \frac{1}{4}$

b) $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x^2} - \sqrt{2-x^2}}{x^2} = ?$

$$\lim_{x \to 0} \frac{\left(\sqrt{2+x^2} - \sqrt{2-x^2}\right)}{x^2} \cdot \frac{\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)}{\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)}$$

$$= \frac{(2+x^2) - (2-x^2)}{x^2\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)} = \frac{\cancel{2} + \cancel{x^2} - \cancel{2} + x^2}{\cancel{x^2}\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)} = \frac{2}{\sqrt{x^2+2} + \sqrt{2-x^2}}$$

$$= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

c) $\lim\limits_{t \to 0} \dfrac{t}{\sqrt{4+t} - \sqrt{4-t}} = ?$

$$\lim_{t \to 0} \frac{t}{\sqrt{4+t} - \sqrt{4-t}} \cdot \frac{\sqrt{4+t} + \sqrt{4-t}}{\sqrt{4+t} + \sqrt{4-t}} = \frac{t\left(\sqrt{4+t} + \sqrt{4-t}\right)}{\cancel{4} + t - \cancel{4} + t}$$

$$= \frac{1}{2} \cdot \left(\sqrt{4+t} + \sqrt{4-t}\right)$$

$$= \underline{\underline{2}}$$

Ex: The $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(h)}{h}$ occurs frequently in the study of calculus. Evaluate this limit for the fonctions f.

1) $f(x) = x^3$

$$= \frac{(x+h)^3 - x^3}{h} = \frac{3x^2 \cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2}}{\cancel{h}} = 3x^2 + 3x\cancel{h} + \cancel{h}^2$$

$$= \underline{\underline{3x^2}}$$

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(2) $f(x) = \frac{1}{x^2}$

$$= \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \frac{x^2 - (x+h)^2}{h \cdot x^2 (x+h)^2}$$

$$= \frac{(x - x - h)(x + x + h)}{h \cdot x^2 \cdot (x+h)^2} = \frac{-\cancel{h} \cdot (2x+h)}{\cancel{h} \cdot x^2 (x+h)^2}$$

$$= \frac{-(2x+h)}{x^2 (x+h)^2}$$

$$= \frac{-2x}{x^2 \cdot x^2} = \frac{-2}{x^3}$$

(3) $f = \frac{1}{\sqrt{x}}$

$$= \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x})(\sqrt{x+h})}$$

$$= \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x})(\sqrt{x+h})} \cdot \frac{(\sqrt{x} + \sqrt{x+h})}{\sqrt{x} + \sqrt{x+h}}$$

$$= \frac{\overset{-h}{x - (x+h)}}{\cancel{h} \cdot \sqrt{x} \cdot \sqrt{x+h} \cdot (\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x} \cdot \sqrt{x} \cdot (\sqrt{x} + \sqrt{x})}$$

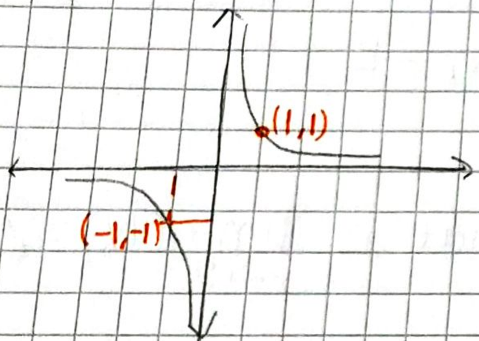
$$= \frac{-1}{2\sqrt{x} \cdot x} = \frac{-1}{2x^{3/2}}$$

(4) $f(x) = \sqrt{x}$

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

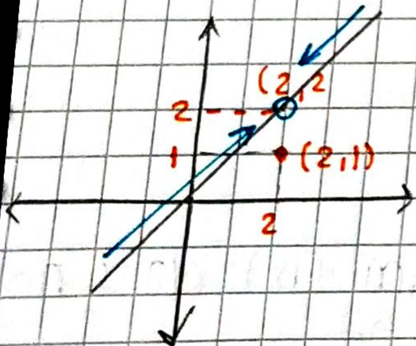
$$= \frac{\cancel{x} + h - \cancel{x}}{\cancel{h} \cdot (\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

* A function f may be defined on both sides of $x = a$ but still not have a limit at $x = a$.



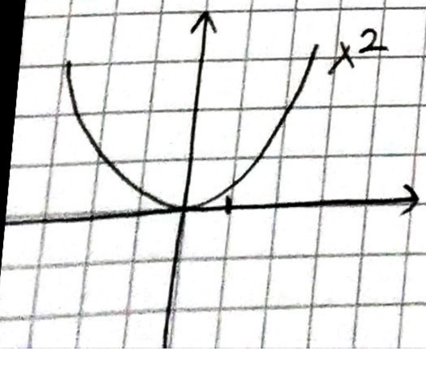
$\lim\limits_{x \to 0} \frac{1}{x}$ does not exist.

Ex: Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



$g(2) = 1$ $\qquad \lim\limits_{x \to 2} g(x) = 2$

Ex: Let $f(x) = \begin{cases} x^2 \Rightarrow \text{if } x \neq 1 \\ 0, \Rightarrow \text{if } x = 1 \end{cases}$



$f(1) = 0$

$\lim\limits_{x \to 1} f(x) = 1$

$\boxed{\text{Definition}}$: If $f(x)$ is defined on some interval (h, a) extending to the left of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the left of a and close enough to a, then we say $f(x)$ has left limit L at $x = a$, and we write:

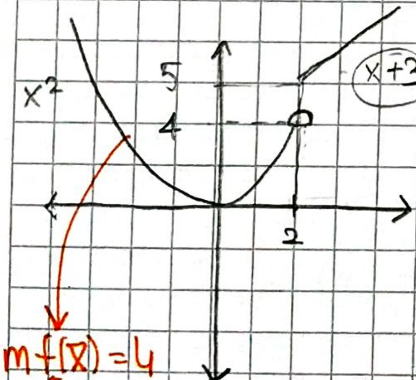
$$\lim_{x \to a^-} f(x) = L$$

the same rules applies to <u>right</u> limit L

$$\lim_{x \to a^+} f(x) = L$$

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

<u>Example</u>: $f(x) = \begin{cases} x^2, & x < 2 \\ x+3, & x \geq 2 \end{cases}$ what is $\lim_{x \to 2} f(x) = ?$



$x+3 \to \lim_{x \to 2^+} f(x) = 5$

$\lim_{x \to 2^-} f(x) = 4$

As a result, $\lim_{x \to 2} f(x)$ does not exist.

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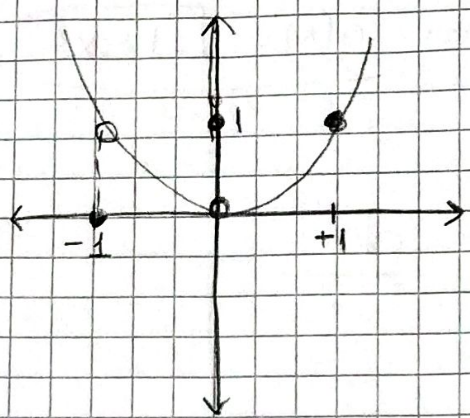
Ex :

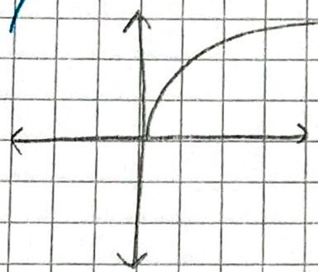


a) $\lim\limits_{x \to -1} f(x) = 1$

b) $\lim\limits_{x \to 0} f(x) = 0$

c) $\lim\limits_{x \to 1} f(x) = 1$

Ex: $\lim\limits_{x \to 0} \sqrt{x} = ?$



$\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = 1$

Ex: If $f(x) = \dfrac{|x-2|}{x^2+x-6}$, find: $\lim\limits_{x \to 2^+} f(x) = ?$ $\dfrac{\cancel{x-2}}{(x+3)\cancel{(x-2)}}$

$= \dfrac{1}{x+3} = \dfrac{1}{5}$

$\lim\limits_{x \to 2^-} f(x) = ?$ $\dfrac{\cancel{2-x}}{(x+3)\cancel{(x-2)}}$

$= -\dfrac{1}{5}$

$\lim\limits_{x \to 2} f(x) = ?$ does not exist

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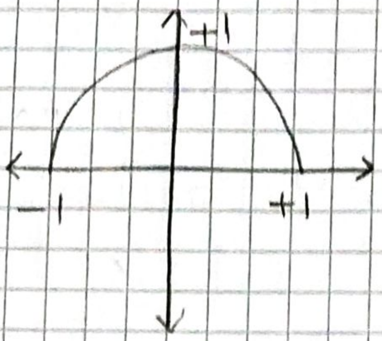
yarım çember formülü

Ex: what one-sided limit does $g(x) = \sqrt{1-x^2}$ have at $x=-1$ and $x=1$?



$\lim_{x \to 1^+} f(x) =$ dne

$\lim_{x \to 1^-} f(x) = 0$

$\lim_{x \to -1^+} f(x) = 0$

$\lim_{x \to -1^-} f(x) =$ dne

Ex:

$$f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x \ge 0 \end{cases}$$

a) $\lim_{x \to -1^-} f(x) = ?$

$x - 1 = (-1) - 1 = -2$

b) $\lim_{x \to -1^+} f(x) = ?$

$x^2+1 \to (-1)^2+1$

$= 2$ //

c) $\lim_{x \to 0^+} f(x) = ?$

$(x+\pi)^2 = \pi^2$

d) $\lim_{x \to 0^-} f(x) = ?$

$(x^2+1) = 1$

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Ex : Evaluate the limits or explain why they don't exist.

a) $\lim_{x \to 2} \frac{\sqrt{4-4x+x^2}}{x-2}$

$= \frac{\sqrt{(x-2)^2}}{x-2} = \frac{|x-2|}{x-2}$

$\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2^-} f(x) = -1$

It does not exist

Ex : b) $\lim_{x \to 0} \frac{|3x-1| - |3x+1|}{x}$

$\lim_{x \to 0^+} \frac{|3x-1| - |3x+1|}{x}$

$\frac{(1-3x) - (3x+1)}{x}$

$\frac{-6x}{x} = -6$

$\lim_{x \to 0^-} \frac{|3x-1| - |3x+1|}{x}$

$\frac{(1-3x) - (3x+1)}{x}$

$\frac{-6x}{x} = -6$

0

$\lim_{x \to 3} \frac{|5-2x| - |x-2|}{|x-5| - |3x-7|} \Rightarrow \lim_{x \to 3} \frac{(2x-5) - (x-2)}{(5-x) - (3x-7)} = \frac{2x-5-x+2}{5-x-3x+7}$

$= \frac{x-3}{12-4x} = -\frac{1}{4}$

* d) $\lim\limits_{x \to 1} \dfrac{\sqrt{3+x} - 2}{\sqrt[3]{7+x} - 2} = ?$ $\sqrt[3]{7+x} = (7+x)^{\frac{1}{3}}$

$\Rightarrow a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$x+7$ 2^3 $(7+x)^{\frac{1}{3}}$ 2

eşlenik

$$\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{\sqrt[3]{7+x} - 2} \cdot \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2} \cdot \frac{\left((x+7)^{2/3} + 2.(x+7)^{1/3} + 4\right)}{\left((x+7)^{2/3} + 2(x+7)^{1/3} + 4\right)}$$

$a^3 - b^3$ için

$$\lim_{x \to 1} \frac{\cancel{(x+3-4)}^{\,x-1}\left((x+7)^{2/3} + 2.(x+7)^{1/3} + 4\right)}{\left(\sqrt{x+3} + 2\right)\underbrace{\cancel{(7+x-8)}}_{x-1}}$$

$$\lim_{x \to 1} \frac{\left((x+7)^{2/3} + 2(x+7)^{1/3} + 4\right)}{\left(\sqrt{x+3} + 2\right)} = \frac{8^{2/3} + 2.8^{1/3} + 4}{4} = \underline{\underline{3}}$$

Rules for Calculating Limits

$\lim\limits_{x \to a} f(x) = L$ $\lim\limits_{x \to a} g(x) = m$ $k = \text{constant}$

1) sum : $\lim\limits_{x \to a} \left[f(x) + g(x) \right] = L + m$

2) difference : $\lim\limits_{x \to a} \left[f(x) - g(x) \right] = L - m$

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3) product : $\lim_{x \to a} f(x) \cdot g(x) = L \cdot m$

4) multiple : $k \cdot \lim_{x \to a} f(x) = k \cdot L$

5) quotient : $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{m}$, if $m \neq 0$

* If n is an integer, and n is positive integer, then

6) powers : $\lim_{x \to a} \left[f(x)\right]^{m/n} = L^{m/n}$, provided $L > 0$ if n is even and $L \neq 0$ if $m < 0$.

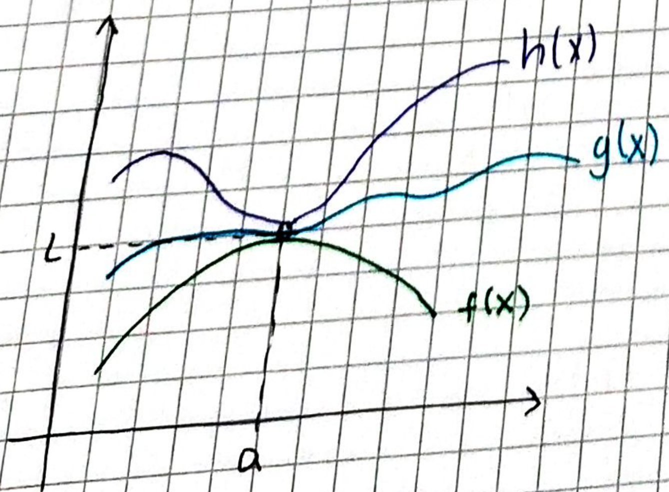
LIMITS OF FUNCTIONS

The squeeze Theorem

* Suppose that $f(x) \leq g(x) \leq h(x)$ holds for all x in some open interval containing "a" except possibly at $x = a$ itself.

Suppose also that,

$\lim_{x \to a} g(x) = L$ also, Similar statements hold for left and right limits.



Example: Given that $3-x^2 \le u(x) \le 3+x^2$ for all $x \ne 0$

find $\lim\limits_{x\to 0} u(x)$.

$\underbrace{\lim\limits_{x\to 0}(3-x^2)=3 \quad \text{and} \quad \lim\limits_{x\to 0}(3+x^2)=3}$

$\Rightarrow$ Theorem iplies that $\lim\limits_{x\to 0} u(x)=3$

Example: If $2-x^2 \le g(x) \le 2\cos x$ for all x, find $\lim\limits_{x\to 0} g(x)$

$\underbrace{\lim\limits_{x\to 0} 2-x^2 = 2 \qquad \lim\limits_{x\to 0} 2\cos x = 2}$

We have $\lim\limits_{x\to 0} g(x) = 2$ by squeeze theorem

Example: $\lim\limits_{x\to 0} \underline{(x^4+x^2)}.\sin\left(\frac{1}{x}\right) = ?$ → $f(x)$ → eşlenik al.

$-x^4-x^2 \le f(x) \le x^4+x^2$ $\qquad -1 \le \sin\left(\frac{1}{x}\right) \le 1$

$\lim\limits_{x\to 0} = 0 \qquad 0$

$\lim\limits_{x\to 0} (x^4+x^2).\sin\left(\frac{1}{x}\right) = 0$

0

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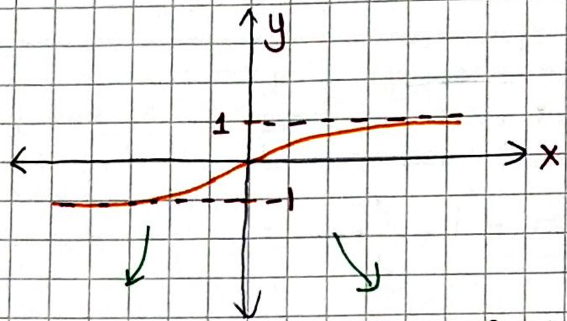
$f(x) = \dfrac{x}{\sqrt{x^2+1}}$ The graph of this.



$\lim_{x \to -\infty} f(x) = -1$ $\lim_{x \to \infty} f(x) = 1$

DEFINITION :

* Limits at infinity and negative intinity (informal definition)

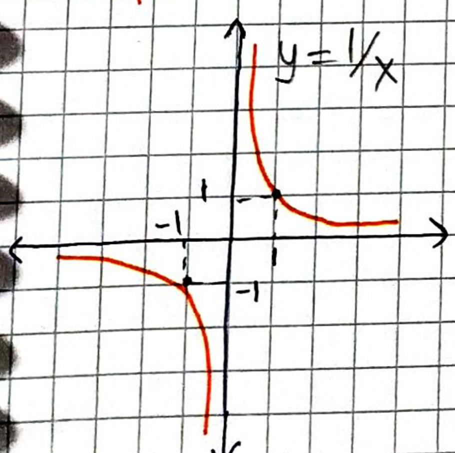
⇒ If the function f is defined on an interval (a, ∞) and if we can ensure that f(x) is as close as we want to the number L by taking x large enough then we say

$\lim_{x \to \infty} f(x) = L$

$\parallel$ $(-\infty, b) \Rightarrow m$ by taking negative.

$\lim_{x \to (-\infty)} f(x) = m$

Example:



$\lim_{x \to \infty} \frac{1}{x} = 0 = \lim_{x \to -\infty} \frac{1}{x} = 0$

$\lim_{x \to \pm\infty} f(x) \Rightarrow \lim_{x \to \pm\infty} \left(\frac{1}{x}\right) \Rightarrow 0$

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Example: Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x)=\frac{x}{\sqrt{x^2+1}}$?

$$\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)}$$

$$\hookrightarrow \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{|x|}\cdot\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1$$

$$\hookrightarrow \lim_{x\to-\infty} \frac{x}{|x|\cdot\sqrt{1+\frac{1}{x^2}}} = -1\cdot\frac{1}{1} = -1$$

Example: $\lim_{x\to\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$ $\rightarrow$ $\frac{x^2}{x}$ çıkarmamız gerek

$$\frac{x\left(2-\frac{1}{x}\right)}{\sqrt{x^2\left(3+\frac{1}{x}+\frac{1}{x^2}\right)}} = \lim_{x\to\infty}\left(\frac{\cancel{x}\left(2-\frac{1}{x}\right)}{\cancel{|x|}\left(3+\frac{1}{x}+\frac{1}{x^2}\right)}\right)$$

$\rightarrow 1$

$$= \frac{2}{\sqrt{3}}\ (+)$$

$\lim_{x\to-\infty}$ hali nedir?

$$\frac{\cancel{x}\left(2-\frac{1}{x}\right)}{\cancel{|x|}\left(\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}\right)} \rightarrow -\frac{2}{\sqrt{3}}$$

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$$\lim_{x \to -\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right)$$

$$\frac{\left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right) \cdot \left(\sqrt{x^2+2x} + \sqrt{x^2-2x}\right)}{\sqrt{x^2+2x} + \sqrt{x^2-2x}}$$

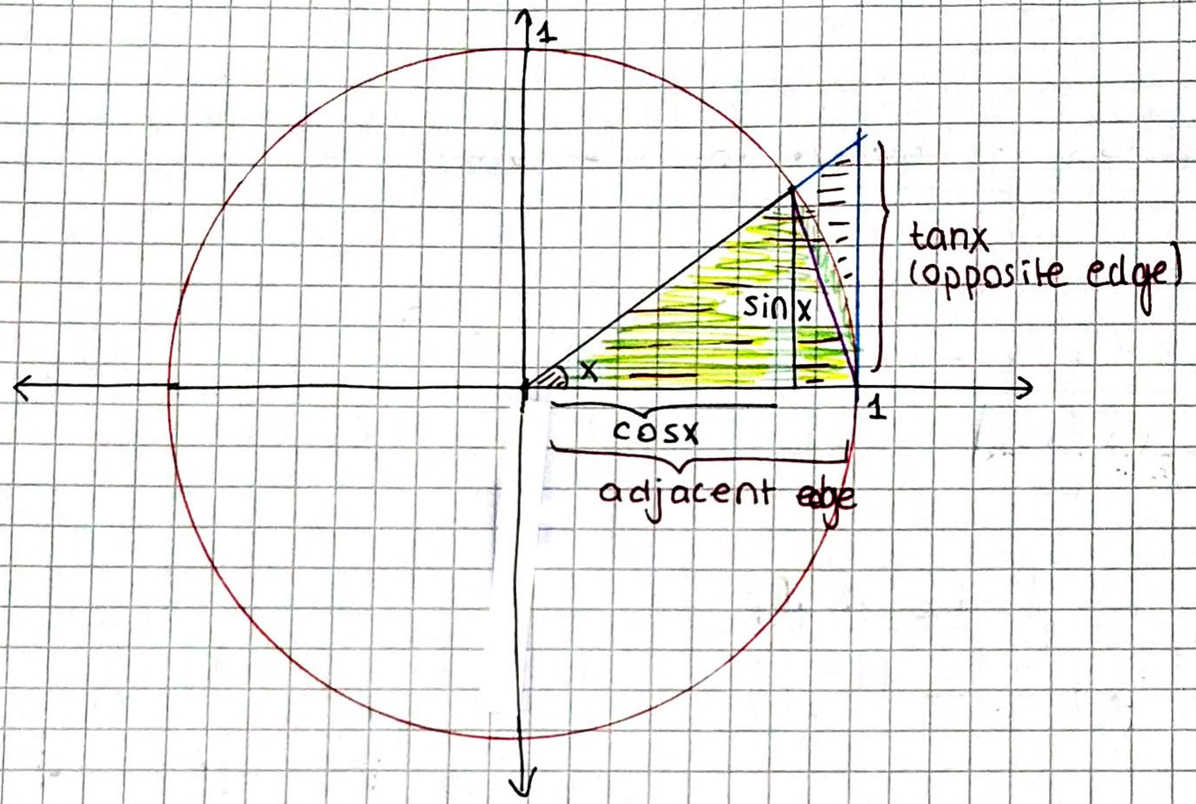
$$= \frac{\cancel{x^2} + 2x - (\cancel{x^2} - 2x)}{\sqrt{x^2+2x} + \sqrt{x^2-2x}} = \frac{4x}{\sqrt{x^2+2x} + \sqrt{x^2-2x}}$$

$$\frac{4x}{\sqrt{x^2\left(1+\frac{2}{x}\right)} + \sqrt{x^2\left(1-\frac{2}{x}\right)}} = \frac{4x}{|x|\sqrt{\frac{2}{x}+1} + |x|\sqrt{1-\frac{2}{x}}}$$

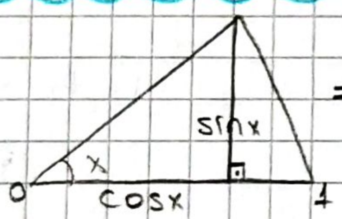
$$= \frac{\cancel{4x}}{\cancel{|x|}\left(\sqrt{\frac{2}{x}+1} + \sqrt{1-\frac{2}{x}}\right)}$$

$$\frac{-4}{1+1} = -2$$

week-3



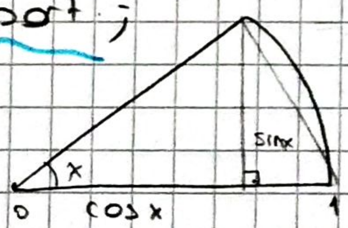
* Area of yellow shaded triangle;



$\Rightarrow A1 = \frac{\sin x \cdot 1}{2}$ → Triangle 1

* We have to find area of triangular region correspond-ing to angle x

→ green shaded part;



Relationship

→ If we have 2π angle ⟶ area: π

$$\pi \cdot \frac{2\pi}{2\pi} = \pi \quad \left(x \text{ angle} \longrightarrow \frac{x}{2}\right) \rightarrow A_2$$

$$\pi \cdot \frac{x}{2\pi} = \frac{x}{2}$$

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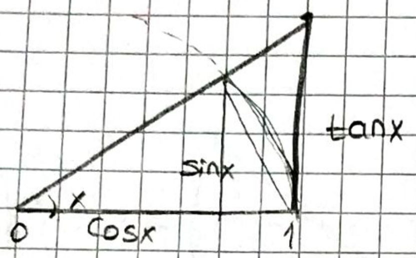
$\rightarrow$ $\tan x = \frac{opp\ x}{adj} = \frac{opp\ x}{1}$

$\rightarrow$ The area of largest (red) triangle;



$$\frac{\tan x \cdot 1}{2} = A_3$$

$\Rightarrow$ $A_1 \leq A_2 \leq A_3$

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}$$

$$\frac{\sin x}{\cancel{2}} \leq \frac{x}{\cancel{2}} \leq \frac{\tan x}{\cancel{2}}$$

$$\Rightarrow \frac{\sin x}{\sin x} \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x}$$

$$\Rightarrow 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

$$\Rightarrow 1 \geq \frac{\sin x}{x} \geq \cos x$$

squezee theorem

$$1 \geq \frac{\sin x}{x} \geq \cancel{\cos x}$$

$$1 \qquad 1 \qquad 1$$

$x \to 0$

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Example:

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \underline{\underline{2}} \Rightarrow \underline{\underline{\text{why}}}: \frac{\sin 2x}{x} = 2 \cdot \underbrace{\frac{\sin 2x}{2x}}_{1} = 2$$

Example:

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^2 \cdot \sin x^2}$$

$$= \frac{\sin^2\left(\frac{x^2}{2}\right) + \cancel{\cos^2\left(\frac{x^2}{2}\right)} - \left[\cancel{\cos^2\left(\frac{x^2}{2}\right)} - \sin^2\left(\frac{x^2}{2}\right)\right]}{x^2 \cdot 2\sin\left(\frac{x^2}{2}\right) \cdot \cos\left(\frac{x^2}{2}\right)}$$

$$= \frac{\cancel{2}\sin^{\cancel{2}}\left(\frac{x^2}{2}\right)}{x^2 \cdot \cancel{2}\cancel{\sin\left(\frac{x^2}{2}\right)} \cdot \cos\left(\frac{x^2}{2}\right)}$$

$$= \frac{\sin\left(\frac{x^2}{2}\right)}{x^2 \cdot \cos\left(\frac{x^2}{2}\right)} = \sin\left(\frac{x^2}{2}\right) \cdot \frac{1}{x^2} \cdot \frac{1}{\cos\left(\frac{x^2}{2}\right)}$$

$$= \frac{1}{2} \cdot \underbrace{\frac{\sin\left(\frac{x^2}{2}\right)}{\frac{x^2}{2}}}_{1} \cdot \underbrace{\frac{1}{\cos\left(\frac{x^2}{2}\right)}}_{1} = 1/2$$

Half Angle Formulations

* $\cos 2x = \cos^2 x - \sin^2 x$
* $\sin 2x = 2\sin x \cdot \cos x$
* $\cos x^2 = \cos\left(2 \cdot \frac{x^2}{2}\right)$ * $\sin x^2 = \sin\left(2 \cdot \frac{x^2}{2}\right)$

* $1 = \cos^2 x + \sin^2 x$
* $1 = \cos^2\left(\frac{x^2}{2}\right) + \sin^2\left(\frac{x^2}{2}\right)$

whatever I want :)

Continuity

* Domain of f is (a,b), then c said to be interior if $a < c < b$.

Definition: Continuity at an interior point

* we say that a function is continuous at an interior point c of its domain f;

$$\lim_{x \to c} f(x) = f(c)$$

* If either $\lim_{x \to c} f(x)$ fails to exist or it exists but it is not equal to $f(c)$. Then we'll say that f is discontinuous at c.

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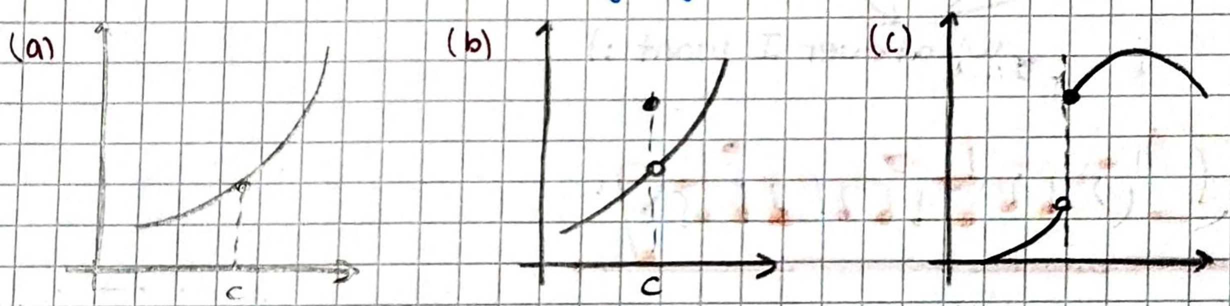
* a function f is "continuous" at a point $x=a$ if;

a) $f(x)$ is defined → a is in the domain of f

b) $\lim_{x \to a} f(x)$ exists → both one-sided limits exist and are equal to L

c) $\lim_{x \to a} f(x) = f(a) = L$ → ✓

* In graphical terms, f is continuous at an interior point c of its domain if its graph has no break in it at the point $(c, f(c))$; in other words, if you can draw the graph through that point without lifting your pen from the paper.



(a) f is continuous at c.

(b) f is discontinuous at c. because $\lim_{x \to c} f(x) \neq f(c)$

(c) f is discontinuous at c because $\lim_{x \to c} f(x)$ does not exist.

* Right and Left Continuity

- we say that f is right continuous at c if $\lim_{x \to c^+} f(x) = f(c)$

left continuous at c if $\lim_{x \to c^-} f(x) = f(c)$

- f is continuous at c if and only if it is both right cont. and left cont. at c.

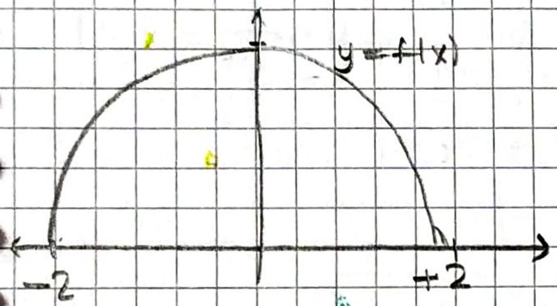
* $f(x) = \sqrt{x}$

$\Rightarrow$ we can write; $\underbrace{\lim_{x \to 0^+} \sqrt{x} = 0 = f(0)}_{\text{right continuous at the point } = 0}$

** If f is continuous at a left endpoint $\longrightarrow$ right continuous there

at a right endpoint $\longrightarrow$ left ~~each~~ continuous there.

<u>Example</u>; The function $f(x) = \sqrt{4 - x^2}$ has domain $[-2, 2]$



there are: · no jumping
· not defined

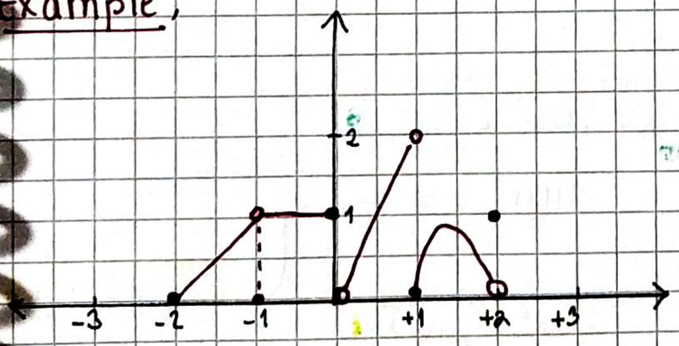
$\lim_{x \to 2^+} f(x) = \text{dne}$

$\lim_{x \to 2^-} f(x) = f(2)$) $2 \rightarrow$ left cont.

$\lim_{x \to -2^+} f(x) = f(-2) \rightarrow -2 \rightarrow$ right cont.

$\lim_{x \to -2^-} f(x) = \text{dne}$

<u>Example</u>;



① <u>at -2:</u> $\lim_{x \to -2^-} \rightarrow \text{dne}$
$\lim_{x \to -2^+} \rightarrow 0 = f(-2)$ } right cont.

② <u>at -1:</u> $\lim_{x \to -1^-} = \lim_{x \to -1^+} \neq f(-1)$

↳ discontinuous

③ <u>at 0:</u> $\lim_{x \to 0^-} = 1$ $\quad \lim_{x \to 0^+} = 0$

$f(0) = 1$

↳ left cont.

④ <u>at 1</u>

$\lim_{x \to 1^-} \rightarrow 2$

$\lim_{x \to 1^+} \rightarrow 0$

$f(1) = 0$

} right cont.

⑤ <u>at 2</u>;

$\lim_{x \to 2^-} = 0$

$\lim_{x \to 2^+} = \text{dne}$

$f(2) = 1$

discont.

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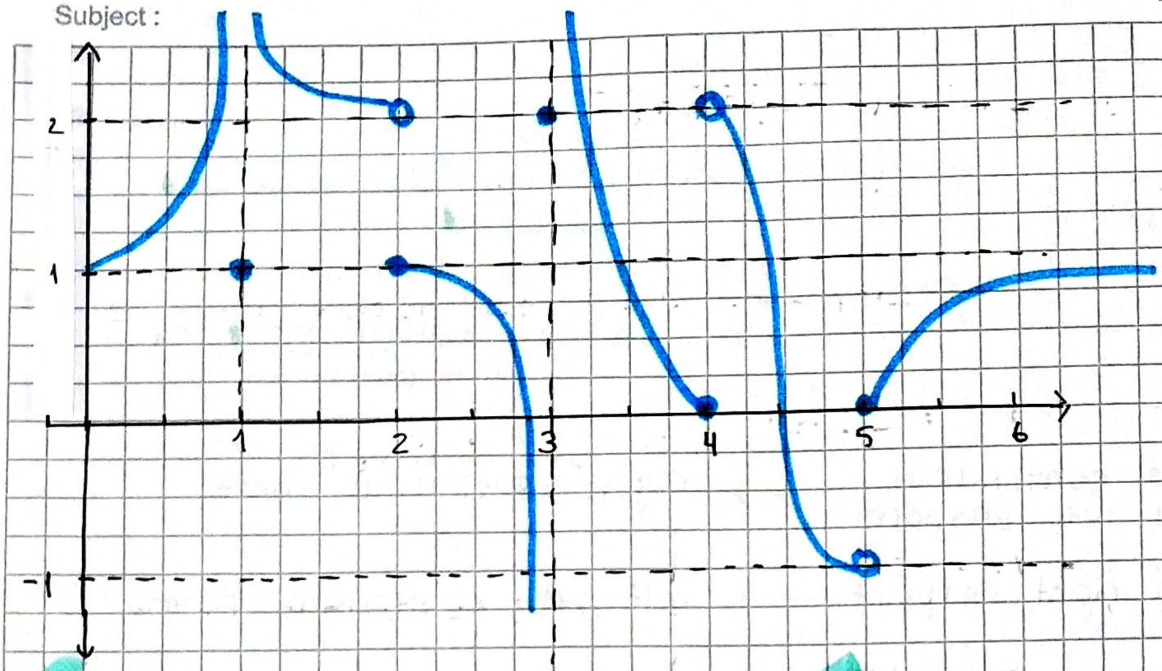

① $\boxed{\text{at } x=0}$

$\lim_{x \to 0^+} = 1$

$f(0) = 1$

Right continuous.

② $\boxed{\text{at } x=1}$

$\lim_{x \to 1^-} = \lim_{x \to 1^+} = \infty \neq f(1)$

↳ discontinuous

③ $\boxed{\text{at } x=2}$

$\lim_{x \to 2^-} = 2$

$\lim_{x \to 2^+} = 1$

$f(2) = 1$

Right continuous

④ $\boxed{\text{at } x=3}$

$\lim_{x \to 3^-} = -\infty$

$\lim_{x \to 3^+} = +\infty$

$f(3) = 2$

discontinuous

⑤ $\boxed{\text{at } x=4}$

$\lim_{x \to 4^+} = 2$

$\lim_{x \to 4^-} = 0$

$f(4) = 0$

Left continuous

⑥ $\boxed{\text{at } x=5}$

$\lim_{x \to 5^-} = -1$

$\lim_{x \to 5^+} = 0$

$f(5) = 0$

Right cont.

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Example : let $f(x) = \begin{cases} \dfrac{\sqrt{9x^4 + x^2}}{5x^2 + 3x + 1} & , \text{ if } x \le 0 \\ x & , \text{ if } x > 0 \end{cases}$ Is f continuous at $x = 0$?

* if $\lim_{x \to 0} f(x) = f(0) \longrightarrow f(x)$ is continuous at $x = 0$.

** $\lim_{x \to 0^-} f(x) = \frac{0}{1} = 0$

$\lim_{x \to 0^+} f(x) = 0$

$f(0) = 0$

$\lim_{x \to 0} f(x) = 0$ and f is continuous at $x = 0$

Example : Let $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & , \text{ if } x < 3 \\ cx^2 + 10 & , \text{ if } x \ge 3 \end{cases}$ • find the value of c so that f(x) is continuous at $x = 3$.

* f(x) is continuous at $x = 3$ if $\lim_{x \to 3} f(x) = f(3)$

** $f(3) = 9c + 10$

$\lim_{x \to 3^-} f(x) = x + 3 = 6$

$\lim_{x \to 3^+} f(x) = 9c + 10$

$9c + 10 = 6$

$9c = -4$

$c = -4/9$

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Example: let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + 3 & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}$

• Is f continuous at $x = 0$?

* It is defined at $x = 0 \longrightarrow f(0) = 1$

** $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \Rightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

⇒ Since for all x;

$$\underset{\downarrow \atop 0}{-x^2} \leq \underbrace{x^2 \cdot \sin\left(\frac{1}{x}\right)}_{\text{squeeze theorem}} \leq \underset{\downarrow \atop 0}{x^2}$$

$$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$$

$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$, which implies;

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) + 3 = 3$$

* Since the value of limit DOES NOT EQUAL the value of the function.

* f(x) is NOT CONTINUOUS at $x = 0$

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Example: Find the real numbers a and b such that the following function is continuous for all x.

$$f(x) = \begin{cases} -2\sin x \longrightarrow & x \le -\frac{\pi}{2} \\ a.\sin x + b \longrightarrow & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x \longrightarrow & x \ge \pi/2 \end{cases}$$

* $f(-\pi/2) = -2\,(\underbrace{\sin -\pi/2}_{-1}) = 2$

$\lim_{x \to -\pi/2^-} = 2$ $\qquad \lim_{x \to -\pi/2^+} = a.\sin(-\pi/2) + b$

$= -a + b$

* $f(\pi/2) = \cos(\pi/2)$

$= 0$

$\lim_{x \to \pi/2^-} = a.\sin \underbrace{(\pi/2)}_{1} + b$ $\qquad \lim_{x \to \pi/2^+} = \cos(\pi/2) = 0$

$= a + b$

↳ $2 = b - a$

$0 = a + b$

$+$

$2 = 2b$

$1 = b$

$-1 = a$

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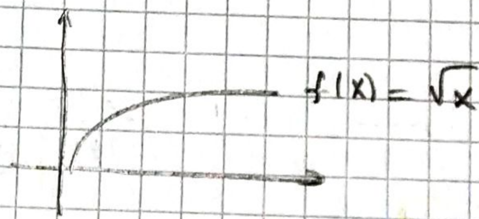
*** Continuity on an Interval

→ we say that function f is continuous on the interval I if it is continous at each point of I. In particular, we will say that f is a continuous function if f is continous at every point of its domain.

Example: The function $f(x) = \sqrt{x}$ is a continuous function. Its domain is $[0, \infty)$



Right cont, 1)

It is continuous at the left endpoint because it is right cont. there. Also f is cont. at every number $c > 0$, since;

$$\lim_{x \to c} \sqrt{x} = \sqrt{c}$$

The following functions are continuous wherever they are defined.

all polynomials / rational functions

and / rational powers

Combining Continuous Functions

* If the functions f and g are both defined on an interval containing c and both are continuous at c, Then the following functions are also continuous at c.

1. the sum $f+g$ and difference $f-g$.
2. the product fg
3. the constant multiple kf, where k is any number.
4. the quotient f/g (provided $g(c) \neq 0$)
5. the nth root $(f(x))^{1/n}$, provided $f(c) > 0$ if n is even.

Continuous Extensions and Removable Discontinuities

* A rational function may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exist, we can define a new function $F(x)$ by,

$$F(x) = \begin{cases} f(x) \rightarrow x \text{ in the domain of } f. \\ L \rightarrow x = c \end{cases}$$

* $F(x)$ is continuous at $c = x$. It is called the continuous extension of $f(x)$ to $x = c$. For rational functions f, continuous extensions are usually found by canceling common factors.

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<u>Example</u>: show that $f(x) = \frac{x^2 - x}{x^2 - 1}$ has a continuous extension to $x = 1$. and find that extension.

** $f(1)$ is not defined.

if $x \neq 1$ we have $f(x) = \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \frac{x}{x+1}$

The function $F(x) = \frac{x}{x+1}$

* $F(x)$ is equal to $f(x)$ for $x \neq 1$, but it is also continuous at $x = 1$ having there the value $1/2$

* $f(x)$ except with no hole at $(1, 1/2)$

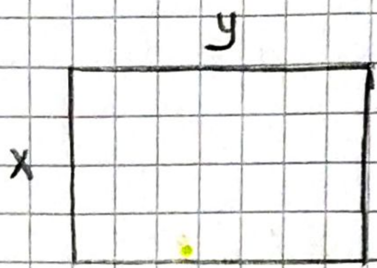
The Max – Min Theorem

* If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then there exist numbers p and q in $[a, b]$ such that for all x in $[a, b]$

$$f(p) \leq f(x) \leq f(q)$$

* Thus f has the absolute minimum value $m = f(p)$, taken on at the point p, and the absolute maximum value $M = f(q)$, taken on at the point q.

Example: What is the largest possible area of a rectangular field that can be enclosed by 200 m of fencing?



Perimeter $= 2x + 2y = P$

Area $= A = x.y$

$P = 200$

$\hookrightarrow x + y = 100$

* $(x-50)^2 = x^2 - 100x + 2500$

$A(x) = 2500 - (x-50)^2$

$A(50) = 2500 \longrightarrow x = 50$

$\searrow y = 50$

Example: The sum of two nonnegative numbers is 8. What is (a) smallest and (b) the largest possible value for the sum of their squares?

$x \geq 0 \quad , \quad y \geq 0$

$S = x^2 + y^2 = x^2 + (8-x)^2$

$= 2x^2 - 16x + 64 = 2(x-4)^2 + 32$

since, $0 \leq x \leq 8$, the maximum value of S occurs at $x=0$ or $x=8$ and is 64

, the minimum value of S occurs at $x=4$ and is 32.

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<u>Example</u> :

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x^n}, & \text{if } x \neq 0 \text{ (n is nonnegative)} \\ \dfrac{x^2+x+2}{\sqrt{x^2+a}}, & \text{if } x = 0 \end{cases}$$

find the values of $\underline{\underline{a}}$ and $\underline{\underline{n}}$ so that the function f is continuous.

$\lim_{x \to 0} f(x) \longrightarrow$ it must exist and be finite.
(var olmak) (sonlu)

$\lim_{x \to 0} \dfrac{\sin x}{x^n} \longrightarrow$ what values of n in this limit are exist, and infinite?

$\underline{\underline{n=0}} \Rightarrow \lim_{x \to 0} \dfrac{\sin x}{x^n} = \dfrac{\sin x}{1} = \sin x = \sin 0 = 0$ //

$\underline{\underline{n=1}} \Rightarrow \lim_{x \to 0} \dfrac{\sin x}{x^n} = \dfrac{\sin x}{x} = 1$ //

$\underline{\underline{n>1}} \Rightarrow \lim_{x \to 0} \dfrac{\sin x}{x^n} =$ infinite.

↳ $\underline{\underline{n=2}}$; $\dfrac{-1}{x^2} \leq \dfrac{\sin x}{x^2} \leq \dfrac{+1}{x^2} \Rightarrow$ approach

* $n=0$, $f(0) \neq 0$
↳ $\lim_{x \to 0} f(x) = 0$ } discontinuity

* $n=1$, $f(0) = \dfrac{2}{\sqrt{a}}$
* $n=1$, $\lim_{x \to 0} f(x) = 1$

} if guarantee that $n=1$, we can say it is continuous and $\underline{\underline{a=4}}$